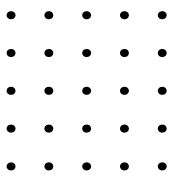
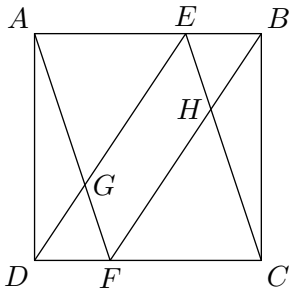
San Diego Math League Euler Olympiad
March 28, 2009

1. A farmer buys a batch of trees, which he wishes to plant in a square grid. For example, if he had 25 trees, then he could plant them as shown below.



However, the farmer finds that he cannot plant his trees in a square grid. If he had 20 more trees, or if he had 39 fewer trees, then he could plant his trees in a square grid. How many trees did the farmer buy?

2. Let $ABCD$ be a square, and let $E$ and $F$ be points on sides $\overline{AB}$ and $\overline{CD}$, respectively, such that $AE : EB = CF : FD = 2 : 1$. Let $G$ be the intersection of $\overline{AF}$ and $\overline{DE}$, and let $H$ be the intersection of $\overline{BF}$ and $\overline{CE}$. Find the ratio of the area of quadrilateral $EGFH$ to the area of square $ABCD$.




3. Find all ordered pairs of positive integers $(a, b)$ such that

$$\frac{1}{a} + \frac{a}{b} + \frac{1}{ab} = 1.$$

4. Sally randomly chooses three different numbers from the set $\{1, 2, \ldots, 14\}$. What is the probability that the sum of her smallest number and her biggest number is at least 15?

5. Let $A = 33 \cdots 3$, where $A$ contains 2009 3s. Let $B = 11 \cdots 1088 \cdots 89$, where $B$ contains 2008 1s and 2008 8s. Prove that $A^2 = B$.